The manifest covariant soliton solutions on noncommutative orbifold T^2/Z_6 and T^2/Z_3

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February 1, 2008

Abstract

In this paper, we construct a closed form of projectors on the integral noncommutative orbifold T^2/Z_6 in terms of elliptic functions by GHS construction. After that, we give a general solution of projectors on T^2/Z_6 and T^2/Z_3 with minimal trace and continuous reduced matrix $M(k, q_0)$.The projectors constructed by us possess symmetry and manifest covariant forms under Z_6 rotation. Since projectors correspond to the soliton solutions of field theory on the noncommutative orbifold, we thus present a series of corresponding manifest covariant soliton solutions.

Keywords: Soliton, Projection operators, Noncommutative orbifold.

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1 Introduction

The idea that the space-time coordinates do not commute is quite old [1]. Indeed, noncommutative geometry has arisen in at least three distinct but closely related contexts in string theory. Witten's open string field theory formulates the interaction of bosonic open strings in the language of noncommutative geometry [2]. Compactification of matrix theory on noncommutative tori was argued to correspond to supergravity with constant background three form tensor field [3]. More generally, it has been realized that noncommutative gauge theory arises in the world-volume theory on D-brane in the presence of a constant background B field in string theory [4]. Until now people have made a lot of contribution to the mathematics and physical application of noncommutative geometry.[5, 6, 7]

Naturally One would like to know what's new that arises from the quantum field theories on noncommutative space. The UV/IR mixing caused by noncommutativity of space-time is one of the intriguing aspects of noncommutative field theory[8, 9]. Noncommutative field theory provides us with a lively description about the quantum Hall effect[10, 11, 12]. The research about the quantum Hall effect concentrates plenty of interest[13]-[25]. As an important object soliton solution always abstracts a lot of concern of string theorists. Although Derrick's theorem forbids solitons in ordinary more than $1 + 1$ dimensions scalar field theory[26], however Gopakumar, Minwalla and Strominger pointed out that there exit soliton solutions in noncommutative scalar field theory[27]. It was soon realized that noncommutative solitons represent D-branes in string field theory with a background B field[28, 29], and many of Sen's conjectures [30, 31] regarding tachyon condensation in string field theory have been beautifully confirmed using properties of noncommutative solitons. Soliton solutions in noncommutative gauge theory were introduced by Polychronakos in [32]. The papers listed in [33, 34] contributed a lot of essential work to the study of solitons in noncommutative gauge theory. The important finding of Gopakumar, Minwalla and Strominger that a projector may correspond to a soliton in the noncommutative field theory in paper [27], shows the significance of study-

ing projection operators in various noncommutative space. Reiffel [35] constructed the complete set of projection operators on the noncommutative torus T^2. On the basis, Boca studied the projection operators on noncommutative orbifold [36] having obtained some beautiful results and the well-known example of projection operator for the case of T^2/Z_4 in terms of the theta function. Martinec and Moore in their important article deeply studied soliton solutions on a wide variety of orbifolds, and the relation between physics and mathematics in this area [37]. Gopakumar, Headrick and Spradlin showed rather a clear method to construct the multi-soliton solution on noncommutative integral torus with generic τ[38]. This approach can be used to construct the projection operators on the integral noncommutative orbifold T^2/Z_N [39].

Some manifest covariant projectors with Z_4 symmetry on noncommutative orbifold T^2/Z_4 were given [36][40]. In [39],we have used the GHS construction to obtain a closed form for the projectors on noncommutative orbifold T^2/Z_6 in terms of theta function. However, Its form is complicated and not explicitly covariant. In this paper, by GHS construction we give the projectors for integral T^2/Z_3 and T^2/Z_6, which is symmetric and manifestly covariant under T^2/Z_6 and T^2/Z_3 rotations . Also, the integration form of this expression include all the projectors with minimal trace and continuous reduced matrices with respect to the variables k and q, just as that in [40].

This paper is organized as following: In Section 2, we briefly review the operators on the noncommutative orbifold T^2/Z_N and GHS construction. In Section 3, we present the explicit and manifest covariant form for the projectors on noncommutative orbifold T^2/Z_6. In the last section, we provide the general covariant projection operators on noncommutative orbifold integral T^2/Z_6 and T^2/Z_3. We conclude this paper with some discussions.

2 Noncommutative Orbifold T^2/Z_N

In this section, we introduce operators on the noncommutative orbifold T^2/Z_N. Let two hermitian operators $\hat{y}_1$ and $\hat{y}_2$ satisfy the following commutation relation:

$$[\hat{y}_1, \hat{y}_2] = i. \tag{1}$$

The operators constituted by the series of $\hat{y}_1$ and $\hat{y}_2$

$$\hat{O} = \sum_{m,n} C_{mn} \hat{y}_1^m \hat{y}_2^n, \quad m, n \in \mathbb{Z} \; and \; m, n \geqslant 0 \tag{2}$$

form a noncommutative plane $\mathbb{R}^2$. All the operators in $\mathbb{R}^2$ which commute with U_1 and U_2 defined by

$$U_1 = e^{-il\hat{y}_2}, \qquad U_2 = e^{il(\tau_2 \hat{y}_1 - \tau_1 \hat{y}_2)}, \tag{3}$$

(where l, τ_1, τ_2 are all real numbers and $l, \tau_2 > 0, \tau = \tau_1 + i\tau_2$), constitute the noncommutative torus T^2. We have

$$\begin{aligned} U_1^{-1} \hat{y}_1 U_1 &= \hat{y}_1 + l, \qquad U_2^{-1} \hat{y}_1 U_2 = \hat{y}_1 + l\tau_1, \\ U_1^{-1} \hat{y}_2 U_1 &= \hat{y}_2, \qquad U_2^{-1} \hat{y}_2 U_2 = \hat{y}_2 + l\tau_2. \end{aligned} \tag{4}$$

The operators U_1 and U_2 are two different wrapping operators around the noncommutative torus and their commutation relation is $U_1 U_2 = U_2 U_1 e^{-2\pi i \frac{l^2 \tau_2}{2\pi}}$. When $A = \frac{l^2 \tau_2}{2\pi}$ is an integer, we have $[U_1, U_2] = 0$ and call the noncommutative torus integral. Define two operators u_1 and u_2:

$$\begin{aligned} u_1 &= e^{-il\hat{y}_2/A}, \qquad u_2 = e^{-il(\tau_2 \hat{y}_1 - \tau_1 \hat{y}_2)/A}, \\ u_1 u_2 &= u_2 u_1 e^{2\pi i/A}, \qquad u_1^A = U_1, \quad u_2^A = U_2^{-1}. \end{aligned} \tag{5}$$

The operators on the noncommutative torus T^2 are composed of the Laurent series of u_1 and u_2,

$$\hat{O}_{T^2} = \sum_{m,n} C'_{mn} u_1^m u_2^n, \tag{6}$$

where $m, n \in \mathbb{Z}$ and C'_{00} is called the trace of the operator. Eq.(6) includes all the operators on the noncommutative torus T^2, satisfying the invariant relation under action of $\{U_i\} : U_i^{-1} \hat{O}_{T^2} U_i = \hat{O}_{T^2}$. We may rewrite the equation(6) as

$$\hat{O}_{T^2} = \sum_{s,t=0}^{A-1} u_1^s u_2^t \Psi_{st}(u_1^A, u_2^A), \tag{7}$$

where Ψ_{st} is the coefficient function of the Laurent series of operators u_1^A and u_2^A. We call this formula standard expansion for the operator on the noncommutative torus T^2. The trace of the operator is the constant term's coefficient of Ψ_{00}. Next we introduce rotation R in noncommutative space $\mathbb{R}^2$,

$$R(\theta) = e^{-i\theta \frac{\hat{y}_1^2 + \hat{y}_2^2}{2} + i\frac{\theta}{2}} \tag{8}$$

with

$$R^{-1} \hat{y}_1 R = \cos\theta \hat{y}_1 + \sin\theta \hat{y}_2, \tag{9}$$

$$R^{-1} \hat{y}_2 R = \cos\theta \hat{y}_2 - \sin\theta \hat{y}_1. \tag{10}$$

Assume $\tau = \tau_1 + i\tau_2 = e^{2\pi i/N}$, $\theta = 2\pi/N (N \in \mathbb{Z})$. Define $R_N \equiv R(2\pi/N)$. Then $U'_i \equiv R_N^{-1} U_i R_N$ can be expressed by monomial of $\{U_i\}$ and their inverses for $A = 2, 3, 4, 6$. For these cases we may introduce the orbifold T^2/Z_N[36, 37]. We call the operators invariant under rotation R_N on the noncommutative torus as operators on noncommutative orbifold T^2/Z_N. We can also realize these operators in Fock space. Introduce

$$a = \frac{\hat{y}_2 - i\hat{y}_1}{\sqrt{2}}, \qquad a^+ = \frac{\hat{y}_2 + i\hat{y}_1}{\sqrt{2}}, \tag{11}$$

then

$$\begin{aligned} [a, a^+] &= 1, \\ R &= e^{-i\theta a^+ a}. \end{aligned} \tag{12}$$

From the above discussion, we know that the operators U_1 and U_2 commute with each other on the integral torus T^2 when A is an integer. So we can introduce a complete set

of their common eigenstates, namely $|k,q\rangle$ representation [38, 41, 42]

$$|k,q\rangle = \sqrt{\frac{l}{2\pi}} e^{-i\tau_1 \hat{y}_2^2/2\tau_2} \sum_j e^{ijkl} |q+jl\rangle, \tag{13}$$

where the ket on the right is the $\hat{y}_1$ eigenstate. We have

$$U_1 |k,q\rangle = e^{-ilk} |k,q\rangle, \qquad U_2 |k,q\rangle = e^{il\tau_2 q} |k,q\rangle = e^{2\pi i q A/l} |k,q\rangle, \tag{14}$$

$$id = \int_0^{\frac{2\pi}{l}} dk \int_0^l dq \, |k,q\rangle \langle k,q|. \tag{15}$$

It also satisfies

$$|k,q\rangle = |k+\frac{2\pi}{l}, q\rangle = e^{ilk}|k,q+l\rangle. \tag{16}$$

$$u_1 |k,q\rangle = \left| k, q+\frac{l}{A} \right\rangle, \qquad u_2 |k,q\rangle = e^{-il\tau_2 q/A} |k,q\rangle = e^{-2\pi i q/l} |k,q\rangle \tag{17}$$

Consider Eq. (7), namely the standard expansion of operators on T^2 we have

$$\Psi_{st}(u_1^A, u_2^A) |k,q\rangle = \Psi_{st}(e^{-ilk}, e^{-2\pi i q A/l}) |k,q\rangle \equiv \tilde{\psi}_{st}(k,q) |k,q\rangle, \tag{18}$$

where $\tilde{\psi}_{st}$ is function of the independent variables k and q, called symbol function of $\Psi_{st}(u_1^A, u_2^A)$. From (18), we see that the function $\tilde{\psi}_{st}$ is invariant when $q \to q + l/A$ or $k \to k + 2\pi/l$,

$$\tilde{\psi}_{st}(k+\frac{2\pi m}{l}, q) = \tilde{\psi}_{st}(k, q+\frac{ln}{A}) = \tilde{\psi}_{st}(k,q). \quad m,n \in \mathbb{Z} \tag{19}$$

As long as the symbol function is obtained, the operator on the noncommutative torus can be completely determined. Introducing new basis $|k,q_0,n\rangle \equiv |k, q_0 + \frac{ln}{A}\rangle, k \in [0, \frac{2\pi}{l}), q_0 \in [0, \frac{l}{A})$, we have from (15)

$$\sum_{n=0}^{A-1} \int_0^{\frac{2\pi}{l}} dk \int_0^{\frac{l}{A}} dq_0 \left| k, q_0 + \frac{ln}{A} \right\rangle \left\langle k, q_0 + \frac{ln}{A} \right| = id. \tag{20}$$

From the above equation and (17)(19), we see that when any power of the operators u_1 and u_2 acts on the $|k, q_0 + \frac{ln}{A}\rangle$, the result can be expanded in the basis $|k, q_0 + \frac{ln}{A}\rangle$ with

the same k, q_0. So all the operators on the noncommutative torus don't change k and q_0. We have

$$\hat{O}_{T^2}\left|k, q_0 + \frac{ln}{A}\right\rangle = \sum_{n'} M^O(k, q_0)_{n'n}\left|k, q_0 + \frac{ln'}{A}\right\rangle. \tag{21}$$

Thus, for every k and q_0 we get an $A \times A$ matrix, called reduced matrix $M^o(k, q_0)$. We have

$$\hat{A}\hat{B}\left|k, q_0 + \frac{ln}{A}\right\rangle = \sum_{n'} \left(M^A(k, q_0)M^B(k, q_0)\right)_{n'n}\left|k, q_0 + \frac{ln'}{A}\right\rangle. \tag{22}$$

For the projection operator on torus T^2,

$$P\left|k, q_0 + \frac{ln}{A}\right\rangle = \sum_{n'} M(k, q_0)_{n'n}\left|k, q_0 + \frac{ln'}{A}\right\rangle. \tag{23}$$

It is easy to find the sufficient and necessary condition for $P^2 = P$ from (22) [39]

$$M(k, q_0)^2 = M(k, q_0). \tag{24}$$

When T^2 satisfies Z_N symmetry, since after R_N rotation U_i' can be expressed by monomial of $\{U_i\}$ and their inverses, the state vector $R_N|k, q_0 + \frac{ln}{A}\rangle$ is still the common eigenstate of the operators U_1 and U_2. With the completeness of $\{\left|k, q + \frac{ls}{A}\right\rangle\}$, and to consider the eigenvalues of U_i in the kq representation, this vector can be expanded in the basis $\{\left|k', q' + \frac{ls'}{A}\right\rangle\}$

$$R_N|k, q_0 + \frac{ln}{A}\rangle = \sum_{n'} A(k, q_0)_{n'n}|k', q_0' + \frac{ln'}{A}\rangle, \tag{25}$$

where $k' \in [0, 2\pi/l), q' \in [0, l/A)$ are definite[39]. Equation (25) gives

$$R_N^{-1}\left|k', q_0' + \frac{ln'}{A}\right\rangle = \sum_{n"} A^{-1}(k, q_0)_{n"n'}\left|k, q_0 + \frac{ln"}{A}\right\rangle. \tag{26}$$

We can get the expression for the relation between k',q_0' and k, q_0, The mapping W : $(k, q_0) \longrightarrow (k', q_0'), W^N = id$, is essentially a linear relation, and area-preserving. By this fact and since R_N is unitary, we conclude that the matrix A is an unitary matrix[39]

$$A^*(k, q_0)_{nn'} = A^{-1}(k, q_0)_{n'n}. \tag{27}$$

Since the projector on the noncommutative orbifold T^2/Z_N satisfies $R_N^{-1}PR_N = P$, then from (23)(25)(26) one obtains

$$R_N^{-1}PR_N|k,q_0+\frac{ln}{A}\rangle = \sum_{n"}[A^{-1}(k,q_0)M(k',q_0')A(k,q_0)]_{n"n}|k,q_0+\frac{ln"}{A}\rangle, \tag{28}$$

which should be equal to :

$$P|k,q_0+\frac{ln}{A}\rangle = \sum_{n"}M(k,q_0)_{n"n}|k,q_0+\frac{ln"}{A}\rangle. \tag{29}$$

So, we have

$$M(k',q_0') = A(k,q_0)M(k,q_0)A^{-1}(k,q_0) \tag{30}$$

Thus the sufficient and necessary condition for the redued matrix of a projector on noncommutative orbifold T^2/Z_N to satisfy is:

$$\begin{aligned} M(k,q_0)^2 &= M(k,q_0), & (31)\\ M(k',q_0') &= A(k,q_0)M(k,q_0)A^{-1}(k,q_0). & (32) \end{aligned}$$

Next we study the relation between the coefficient function $\tilde{\psi}_{st}(k,q)$ and reduced matrix $M(k,q_0)$. Due to (17)(18)(19) and (23)we have

$$\begin{aligned} P|k,q_0+\frac{ln}{A}\rangle &= \sum_{s,t}u_1^s u_2^t \Psi_{st}(u_1^A,u_2^A)|k,q_0+\frac{ln}{A}\rangle \\ &= \sum_{s,t}e^{-2\pi i(q_0/l+n/A)t}\tilde{\psi}_{st}(k,q_0)|k,q_0+\frac{l(n+s)}{A}\rangle \\ &= \sum_{n'}M(k,q_0)_{n'n}|k,q_0+\frac{ln'}{A}\rangle. \end{aligned} \tag{33}$$

From the periodic condition of $|k,q\rangle$ (16), for $n+s<A$ case, we have

$$M(k,q_0)_{n+s,n} = \sum_{t=0}^{A-1}e^{-2\pi i(q_0/l+n/A)t}\tilde{\psi}_{st}(k,q_0), \tag{34}$$

and for $n+s\geq A$ case, we have

$$M(k,q_0)_{n+s-A,n} = \sum_{t=0}^{A-1}e^{-2\pi i(q_0/l+n/A)t}\tilde{\psi}_{st}(k,q_0)e^{-ilk}. \tag{35}$$

Setting

$$M(k,q_0)_{n+s,n} = M(k,q_0)_{n+s-A,n}e^{ilk}, \tag{36}$$

we can uniformly write as:

$$M(k,q_0)_{n+s,n} = \sum_{t=0}^{A-1} e^{-2\pi i(q_0/l+n/A)t}\tilde{\psi}_{st}(k,q_0) \tag{37}$$

and have

$$\tilde{\psi}_{st}(k,q_0) = \frac{1}{A}\sum_{r=0}^{A-1} M(k,q_0)_{r+s,r}e^{2\pi i(q_0/l+r/A)t}. \tag{38}$$

Eq.(37) and (38) is the relation between $\tilde{\psi}_{st}$ and the elements of reduced matrix M.

We set (This is the GHS construction)[38]

$$M(k,q_0)_{nn'} = \frac{\langle k, q_0+\frac{ln}{A}|\phi_1\rangle\langle\phi_2|k,q_0+\frac{ln'}{A}\rangle}{\sum_{n"}\langle k,q_0+\frac{ln"}{A}|\phi_1\rangle\langle\phi_2|k,q_0+\frac{ln"}{A}\rangle}. \tag{39}$$

It satisfies (31) and as long as $R|\phi_j\rangle = e^{i\alpha_j}|\phi_j\rangle$, it also satisfies (32)[40]. Notice that this equation satisfies (36). We then have

$$\begin{aligned}
\tilde{\psi}_{st}(k,q_0) &= \frac{1}{A}\sum_{r=0}^{A-1} M(k,q_0)_{r+s,r}e^{2\pi i(q_0/l+r/A)t} \\
&= \frac{\frac{1}{A}\sum_{r=0}^{A-1}\langle k,q_0+\frac{l(r+s)}{A}|\phi_1\rangle\langle\phi_2|k,q_0+\frac{lr'}{A}\rangle e^{2\pi i(q_0/l+r/A)t}}{\sum_r\langle k,q_0+\frac{lr}{A}|\phi_1\rangle\langle\phi_2|k,q_0+\frac{lr}{A}\rangle} \\
&= \frac{\tilde{f}_{st}(k,q_0)}{A\tilde{f}_{00}(k,q_0)},
\end{aligned} \tag{40}$$

where

$$\tilde{f}_{st}(k,q_0) \equiv \sum_{r=0}^{A-1}\langle k,q_0+\frac{l(r+s)}{A}|\phi_1\rangle\langle\phi_2|k,q_0+\frac{lr}{A}\rangle e^{2\pi i(q_0/l+r/A)t}, \tag{41}$$

with

$$\tilde{f}_{st}(k,q_0) = \tilde{f}_{st}(k,q_0+l/A) = \tilde{f}_{st}(k+2\pi/l,q_0), \tag{42}$$

$$\tilde{f}_{st}(k,q_0) = \tilde{f}_{s+A,t}(k,q_0)e^{-ilk} \tag{43}$$

$$= \tilde{f}_{s,t+A}(k,q_0)e^{-2\pi i q_0 A/l}. \tag{44}$$

Define

$$u = \frac{lk}{2\pi}, \quad v = \frac{q_0}{l}, \tag{45}$$

$$f_{st}(u,Av) \equiv \tilde{f}_{st}(k,q_0). \tag{46}$$

So the function $f_{st}(u, Av)$ is function of the independent variables u and Av with period 1. Similarly define

$$\psi_{st}(u, Av) \equiv \tilde{\psi}_{st}(k, q_0), \tag{47}$$

we have

$$\psi_{st}(u, Av) = \frac{f_{st}(u, Av)}{Af_{00}(u, Av)}. \tag{48}$$

Let

$$\begin{aligned} X &\equiv e^{-ilk} = e^{-2\pi iu}, \\ Y &\equiv e^{-2\pi iqA/l} = e^{-2\pi iAv}. \end{aligned}$$

If we change the variables X and Y into u_1^A and u_2^A respectively in $\psi_{st}(u, Av)$, the standard form (7) of the projection operator can be easily obtained. So the key question is to find out $\tilde{f}_{st}(k, q_0)$. For simplicity, we set

$$|\phi_1\rangle = |\phi_2\rangle = |0\rangle, \quad a|0\rangle = 0, \quad R_N|0\rangle = |0\rangle. \tag{49}$$

After some derivation, we have [38]

$$< \; k, q|0 > \equiv C_0(k, q) = \frac{1}{\sqrt{l}\pi^{1/4}}\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\frac{q}{l} + \frac{\tau k}{l\tau_2}, \frac{\tau}{A})e^{-\frac{\tau}{2i\tau_2}k^2 + ikq} \tag{50}$$

$$= \sqrt{\frac{Ai}{l\tau\sqrt{\pi}}}\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\frac{lk}{2\pi} + \frac{Aq}{l\tau}, -\frac{A}{\tau})e^{-\pi i\frac{Aq^2}{\tau l^2}}, \tag{51}$$

where

$$\theta(z, \tau) \equiv \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau), \tag{52}$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_m e^{\pi i\tau(m+a)^2} e^{2\pi i(m+a)(z+b)}. \tag{53}$$

Define

$$g_{ss'}(u, v) \equiv \langle k, q + \frac{ls}{A}|0\rangle\langle 0|k, q + \frac{ls'}{A}\rangle. \tag{54}$$

Then we get for real u and v,

$$
\begin{aligned}
f_{st}(u, Av) &= \sum_{r=0}^{A-1} g_{s+r,r}(u,v) \times e^{2\pi it(\frac{r}{A}+v)} \\
&= \sum_r \frac{1}{l\sqrt{\pi}} \theta(v + \frac{u\tau + s + r}{A}, \frac{\tau}{A}) \theta(v + \frac{u\tau^* + r}{A}, \frac{-\tau^*}{A}) \times e^{2\pi it(v+\frac{r}{A})} \\
&\quad \times e^{\pi i \frac{\tau - \tau^*}{A} u^2 + 2\pi i \frac{s}{A} u} \\
&= \frac{A}{l|\tau|\sqrt{\pi}} \sum_r \theta(u + \frac{A}{\tau}(v + \frac{s+r}{A}), -\frac{A}{\tau}) \theta(u + \frac{A}{\tau^*}(v + \frac{r}{A}), \frac{A}{\tau^*}) \\
&\quad \times e^{-\pi i \frac{A}{\tau}(v + \frac{s+r}{A})^2 + \pi i \frac{A}{\tau^*}(v + \frac{r}{A})^2} \times e^{2\pi it(v+\frac{r}{A})}. \qquad (55)
\end{aligned}
$$

Then from (55) and properties of theta functions,we have

$$
\begin{aligned}
f_{st}(u+1, Av) &= f_{st}(u, Av+1) = f_{st}(u, Av), \qquad (56) \\
f_{st}(u + A\tau, Av) &= e^{-2\pi i(2u + A(\tau+\tau^*)v + \frac{A}{2}(\tau - \tau^*) + \frac{s}{\tau})} f_{st}(u, Av), \qquad (57) \\
f_{st}(u, A\tau + Av) &= e^{-2\pi i(2Av + (\tau+\tau^*)u + A\frac{\tau - \tau^*}{2} - t\tau)} f_{st}(u, Av). \qquad (58)
\end{aligned}
$$

These are the brief review of the GHS construction the projection operators on noncommutative orbifold T^2/Z_N . In the next section, we will concretely discuss how to construct the manifest covariant projectors on noncommutative orbifold T^2/Z_6.

3 The covariant projectors on noncommutative orbifold T^2/Z_6

In the above section, we reviewed some results for projectors on noncommutative orbifold T^2/Z_N. Boca and we presented some manifest covariant projectors with Z_4 symmetry on noncommutative integral orbifold T^2/Z_4[36][40]. In [39], we have presented a closed form for projectors on the noncommutative orbifold T^2/Z_6 in terms of elliptic function. However, its form is not explicitly covariant. In this section, we are devoted to develop the manifest covariant form for projectors on the noncommutative orbifold T^2/Z_6 by GHS

construction. In the case that $\tau = \tau_6 = e^{\frac{\pi i}{3}}$, we have

$$\begin{aligned} f_{st}(u+1, Av) &= f_{st}(u, Av+1) = f_{st}(u, Av), & (59)\\ f_{st}(u+A\tau, Av) &= e^{-2\pi i(2u+Av+A\tau-\frac{A}{2}+\frac{s}{\tau})} f_{st}(u, Av), & (60)\\ f_{st}(u, A\tau+Av) &= e^{-2\pi i(2Av+u+A\tau-\frac{A}{2}-t\tau)} f_{st}(u, Av). & (61)\end{aligned}$$

From this, it can be proved that $f_{st}(u, Av)$ belongs to a three-dimensional linear space. We can define the basis of this space as

$$\theta(Av+\alpha)\theta(Av+u+\beta)\theta(u+\gamma) \equiv e(u, Av), \qquad (62)$$

such that three linearly independent basis are enough to construct any function satisfying conditions (59)-(61). Here α, β, γ are parameters to be given later and we denote

$$\theta(z) \equiv \theta(z, A\tau) \equiv \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, A\tau)$$

for simplicity. (In the following, theta function without modular parameter means its modular parameter is $A\tau$) We have from (62)

$$\begin{aligned} e(u+1, Av) &= e(u, Av+1) = e(u, Av), & (63)\\ e(u+A\tau, Av) &= e^{-2\pi i(2u+Av+A\tau+\beta+\gamma)} e(u, Av), & (64)\\ e(u, A\tau+Av) &= e^{-2\pi i(u+2Av+A\tau+\alpha+\beta)} e(u, Av). & (65)\end{aligned}$$

Thus we require that

$$\alpha+\beta = -\frac{A}{2} - t\tau, \quad \beta+\gamma = -\frac{A}{2} + \frac{s}{\tau}, \qquad (66)$$

where $\tau = e^{\frac{\pi i}{3}}$. Next, we will consider the covariant property for the projectors. From the definition, it is easy to get for $R = R_6$

$$u_1' = R^{-1}u_1R = u_2^{-1}, \quad u_2' = R^{-1}u_2R = e^{-\pi i/A}u_1u_2, \qquad (67)$$

$$u_1u_2 = e^{2\pi i/A}u_2u_1 \qquad (68)$$

Define

$$c = e^{-\pi i/A},$$

then

$$R^{-1}PR = \sum_{st} u_1^t u_2^{t-s} c^{-2st+t^2} \Psi_{st}(u_2^{-A}, c^{A^2} u_1^A u_2^A). \tag{69}$$

We have from (14)(17) and (45)

$$\begin{aligned} u_1^A |k,q\rangle &= e^{-2\pi i u} |k,q\rangle, \quad u_2^A |k,q\rangle = e^{-2\pi i A v} |k,q\rangle & (70) \\ u_1'^A |k,q\rangle &= e^{-2\pi i(-Av)} |k,q\rangle, \quad u_2'^A |k,q\rangle = e^{-\pi i A} e^{-2\pi i(u+Av)} |k,q\rangle & (71) \end{aligned}$$

From (18) (47) we have

$$\begin{aligned} \Psi_{st}(u_1^A, u_2^A) |k,q\rangle &\equiv \psi_{st}(u, Av) |k,q\rangle \\ R^{-1}\Psi_{st}(u_1^A, u_2^A) R |k,q\rangle &= \Psi_{st}(u_1'^A, u_2'^A) |k,q\rangle & (72) \\ &= \psi_{st}(-Av, -\frac{A}{2} + u + Av) |k,q\rangle. & (73) \end{aligned}$$

That is, the variables u and Av change as

$$u \to -Av, Av \to -\frac{A}{2} + u + Av \tag{74}$$

under the rotation $R = R_6$. Therefore, when $P = R^{-1}PR$, the formulation (7) and (69) demand

$$c^{-2st+t^2} \psi_{st}(-Av, -\frac{A}{2} + u + Av) = \psi_{t,t-s}(u, Av). \tag{75}$$

Notice that $\psi_{00}(u, Av)$ is invariant under rotation R. From (40), we can get

$$\tilde{\psi}_{st}(k, q_0) \equiv \psi_{st}(u, Av) = \frac{f_{st}(u, Av)}{A f_{00}(u, Av)}. \tag{76}$$

If we construct $f_{st}(u, Av)$ satisfying the relation similar to (75) obtaining $\psi_{st}(u, Av)$ by (76), and set $\psi_{st}(u, Av) |kq\rangle = \Psi_{st}(u_1^A, u_2^A) |k,q\rangle$,(see the text after equation (48)) then we can get the projector $P = \sum_{st} u_1^s u_2^t \Psi_{st}(u_1^A, u_2^A)$ which is invariant under rotation R. In the following, we wish to find a set of covariant basis to construct such f_{st}. We write the basis as

$$e(u, Av) = \theta(Av + \alpha)\theta(Av + u + \beta)\theta(u + \gamma).$$

After rotation R, it is turned into

$$\begin{aligned} e'(u, Av) &= \theta(-\frac{A}{2} + u + Av + \alpha)\theta(-\frac{A}{2} + u + \beta)\theta(Av - \gamma) & (77) \\ &= \theta(Av + u + \beta')\theta(u + \gamma')\theta(Av + \alpha'). \end{aligned}$$

Thus the base vector change its parameters under the rotation as

$$\alpha' = -\gamma, \quad \beta' = \alpha - \frac{A}{2}, \quad \gamma' = -\frac{A}{2} + \beta \quad (mod\mathbb{Z}). \tag{78}$$

Now we take the transformation under the light of (75)

$$s' = t, \quad t' = t - s \Rightarrow t = s' \quad , s = s' - t'$$

The covariant base should satisfy

$$e_{st}(-Av, -\frac{A}{2} + u + Av) = e_{t,t-s}(u, Av) = e_{s',t'}(u, Av). \tag{79}$$

We set

$$\begin{aligned} \alpha &= \alpha_{st} = \alpha_1 s + \alpha_2 t + \alpha_3, \\ \beta &= \beta_{st} = \beta_1 s + \beta_2 t + \beta_3, \\ \gamma &= \gamma_{st} = \gamma_1 s + \gamma_2 t + \gamma_3, \end{aligned} \tag{80}$$

here

$$\begin{aligned} \alpha_1 &= \frac{1}{2} + \frac{\sqrt{3}}{6} i, \ \alpha_2 = -\frac{\sqrt{3}}{3} i, \ \alpha_3 = \frac{B}{2}, \\ \beta_1 &= \frac{1}{2} - \frac{\sqrt{3}}{6} i, \ \beta_2 = -\frac{1}{2} - \frac{\sqrt{3}}{6} i, \ \beta_3 = -\frac{A}{2} - \frac{B}{2} \\ \gamma_1 &= -\frac{\sqrt{3}}{3} i, \ \gamma_2 = -\frac{1}{2} + \frac{\sqrt{3}}{6} i, \ \gamma_3 = \frac{B}{2}. \quad B \in \mathbb{Z} \end{aligned} \tag{81}$$

Then (66) is satisfied. From (78)(80) the variables α, β, γ transform as

$$\begin{aligned} \alpha' &= \alpha_1 s' + \alpha_2 t' + \alpha_3 - B = \alpha_{s't'}, \qquad (82) \\ \beta' &= \beta_1 s' + \beta_2 t' + \beta_3 + A = \beta_{s't'}, \\ \gamma' &= \gamma_1 s' + \gamma_2 t' + \gamma_3 - B = \gamma_{s't'} \quad (mod\mathbb{Z}). \end{aligned}$$

Then we have

$$e_{st}(-Av, -\frac{A}{2} + u + Av) = e_{t,t-s}(u, Av) = e_{s',t'}(u, Av). \tag{83}$$

It really satisfies the covariant condition. Setting $B = 0, 1$ in (81), we obtain two linearly independent basis e_0 and e_1 which have the covariant property (83). We verify that $f_{st}(u, Av)$ of (55) can be expanded by just the two basis in the following. We rewrite the $f_{st}(u, Av)$ in (55) by taking

$$\tilde{v}_0 + \frac{\tilde{s}_0}{A} = v + \frac{\tau u}{A} + \frac{s}{A}, \quad \tilde{v}_0 = v + \frac{\tau^* u}{A}. \tag{84}$$

When $\tau = \tau_6 = e^{\frac{2\pi}{6}i}$,

$$\frac{\tilde{s}_0}{A} = \frac{s}{A} + \frac{(2\tau - 1)u}{A} \ . \tag{85}$$

We get $f_{st}(u, Av)$ as follows:

$$\begin{aligned} f_{st}(u, Av) \ &= \ \sum_r \frac{1}{l\sqrt{\pi}} \theta(\tilde{v}_0 + \frac{\tilde{s}_0 + r}{A}, \frac{\tau}{A}) \theta(\tilde{v}_0 + \frac{r}{A}, -\frac{\tau^*}{A}) \times e^{2\pi i t r/A} \\ &\quad \times e^{\pi i \frac{\tau - \tau^*}{A} u^2 + 2\pi i \frac{s}{A} u + 2\pi i t v}. \end{aligned} \tag{86}$$

Expanding the two theta functions involved in (86) by (53) we obtain the following form. (see Appendix A for details)

$$f_{st}(u, Av) = \frac{A}{l\sqrt{\pi}} \sum_{\delta = 0,1} e^{2\pi i \phi'} \theta\left(z', \frac{2\tau - 1}{A}\right) \theta\left(w, A\left(2\tau - 1\right)\right), \tag{87}$$

where

$$\begin{aligned} z' \ &= \ -\delta\tau - \frac{\tau}{A} t + \frac{s}{A} + \frac{2\tau - 1}{A} u, \\ w \ &= \ (\delta A - t)\tau + s + 2Av + u, \\ \phi' \ &= \ \delta A \frac{\tau - 1}{2} + \delta(Av - (\tau - 1)u) + \frac{2\tau - 1}{2A} u^2 + \frac{s}{A} u, \\ &\quad -\frac{\tau}{A} t u - \frac{st}{A} + \frac{\tau}{2A} t^2. \end{aligned} \tag{88}$$

From (55) and (59)-(61), the function $\tilde{f}_{st}(k, q) = f_{st}(u, Av)$ belongs to a three-dimensional space spanned by functions of u and Av, $f_{st}(u, Av)$ can be expanded by the following basis,

$$e_0\left(u, Av\right) \ = \ \theta\left(Av + \alpha\right) \theta\left(Av + u + \beta\right) \theta\left(u + \gamma\right), \tag{89}$$

$$\begin{aligned} e_1\left(u, Av\right) \ &= \ \theta\left(Av + \alpha + \frac{1}{2}\right) \theta\left(Av + u + \beta - \frac{1}{2}\right) \theta\left(u + \gamma + \frac{1}{2}\right), \\ e_2\left(u, Av\right) \ &= \ \theta\left(Av + \alpha + x\right) \theta\left(Av + u + \beta - x\right) \theta\left(u + \gamma + x\right) \qquad 0 < x \ll 1 \end{aligned} \tag{90}$$

where α, β, γ are given in (80)(81) with $B = 0$.We have

$$f_{st} = c_0 e_0 + c_1 e_1 + c_2 e_2 \tag{91}$$

For convenience of derivation we change the arguments as follows:

$$\begin{aligned} Av &= \lambda - \alpha + a, & (92) \\ u &= -\gamma + b, & (93) \end{aligned}$$

where $\lambda = \frac{1}{2}(A\tau + 1)$. Notice $\beta - \alpha - \gamma = \frac{A}{2}$ in the setting of (80)(81) for $B = 0$. Then we have

$$\begin{aligned} e_0 &= \theta(\lambda + a)\,\theta\left(\lambda - \frac{A}{2} + a + b\right)\theta(b), \\ e_1 &= \theta\left(\lambda + a + \frac{1}{2}\right)\theta\left(\lambda - \frac{A}{2} + a + b - \frac{1}{2}\right)\theta\left(b + \frac{1}{2}\right), \\ e_2 &= \theta(\lambda + a + x)\,\theta\left(\lambda - \frac{A}{2} + a + b - x\right)\theta(b + x). \end{aligned} \tag{94}$$

Based on the replacement of arguments given by (92) and (93), we rewrite f_{st} in (87) by variables a and b,

$$\begin{aligned} f_{st}(u, Av)\big|_{v = \frac{\lambda - \alpha + a}{A}}^{u = -\gamma + b} &= \frac{A}{l\sqrt{\pi}} e^{2\pi i \phi_{st}} \sum_{\delta = 0,\ 1} e^{2\pi i \phi_\delta(a,b)} \theta\left(\delta(1 - \tau) + \frac{2\tau - 1}{A} b, \frac{2\tau - 1}{A}\right) \\ &\quad \times \theta\left(\delta A\tau + 2\lambda + 2a + b, A(2\tau - 1)\right), \end{aligned} \tag{95}$$

where

$$\begin{aligned} \phi_{st} &= \frac{\sqrt{3}i}{6A}\left(s^2 + t^2 - st\right) - \frac{1}{2A} st \\ \phi_\delta(a, b) &= \frac{\sqrt{3}i}{2A} b^2 + \delta\left(\delta A \frac{\tau - 1}{2} + \lambda + a + (1 - \tau) b\right) \end{aligned}$$

In order to verify $c_2 = 0$ and determine the coefficients c_0, c_1in (91), consider the case of $b_m = \frac{1-A}{2} + \frac{mA}{2\tau - 1}$, $m \in \mathbb{Z}$ and $a = a_n = \frac{n}{2}, n \in \mathbb{Z}$. Since $\theta\left(n_1 + \frac{1}{2} + \left(n_2 + \frac{1}{2}\right)\tau, \tau\right) = 0, n_1, n_2 \in \mathbb{Z}$, when $\delta = 1$, the first theta function in (95) $\theta\left(\frac{(1-2A)(2\tau-1)}{2A} + m + \frac{1}{2}, \frac{2\tau - 1}{A}\right)$ vanishes. Therefore we have for $\tau = e^{\pi i/3}$.

$$
\begin{aligned}
f_{st}\left(u, Av\right)|^{u=-\gamma+b_m}_{v=\frac{1}{A}(\lambda-\alpha+a_n)} &\equiv f_{mst} \\
&= \frac{A}{l\sqrt{\pi}} e^{2\pi i\phi_{st}+2\pi i\phi_\delta|_{\delta=0}} \theta\left(\frac{(1-A)\left(2\tau-1\right)}{2A}+m, \frac{2\tau-1}{A}\right) \\
&\quad \times\theta\left(A\tau+1+2a_n+\frac{1-A}{2}+\frac{mA}{2\tau-1}, A\left(2\tau-1\right)\right) \\
&= \frac{A}{l\sqrt{\pi}} e^{2\pi i\phi_{st}+2\pi i\phi_\delta|_{\delta=0}} \theta\left(\frac{(1-A)\left(2\tau-1\right)}{2A}+m, \frac{2\tau-1}{A}\right) \\
&\quad \times\theta\left(\frac{A\left(2\tau-1\right)}{2}+\frac{1}{2}-\frac{m}{3}A(2\tau-1)+n, A\left(2\tau-1\right)\right) \quad (96)
\end{aligned}
$$

for $u=-\gamma_{st}+b_m, v=\frac{1}{A}(\lambda-\alpha_{st}+a_n)$, which is independent of $n\in\mathbb{Z}$.

when $m=3p$, $p\in\mathbb{Z}$, the second theta function in the right-hand side of (96) vanishes,

$$f_{mst}=0, \tag{97}$$

when $m\neq 3p$, define

$$f_{st}(u, Av)=f_{mst}\neq 0. \tag{98}$$

Next we check e_0, e_1 and e_2 in these cases, obtaining

- If $a=0$,then $e_0=0$ and

$$e_1^0(m)\equiv e_1|_{a=0}=\theta(\lambda+\frac{1}{2})\theta(\lambda-\frac{m}{3}A(2\tau-1))\theta(b_m+\frac{1}{2});$$

- If $a=-\frac{1}{2}$, then $e_1=0$ and

$$e_0^0(m)\equiv e_0|_{a=-\frac{1}{2}}=\theta(\lambda-\frac{1}{2})\theta(\lambda-\frac{m}{3}A(2\tau-1))\theta(b_m); \tag{99}$$

- When $m=3p, e_0=e_1=0, e_2\neq 0$.

From (91), (97)we can obtain

$$c_2=0 \tag{100}$$

and

$$f_{st}(u, Av)=c_0e_0+c_1e_1. \tag{101}$$

Then we take $m \neq 3p$, giving

$$\begin{aligned} f_{mst} &= c_1 e_1^0(m) \\ f_{mst} &= c_0 e_0^0(m) \end{aligned}$$

$$\Longrightarrow \; c_1 = \frac{f_{mst}}{e_1^0(m)}, c_0 = \frac{f_{mst}}{e_0^0(m)}, \tag{102}$$

So we have

$$f_{st} = f_{mst}\left(\frac{e_0}{e_0^0(m)} + \frac{e_1}{e_1^0(m)}\right). \tag{103}$$

Besides, we have

$$\frac{f_{mst}}{f_{m00}} = e^{2\pi i\phi_{st}},$$

and the ratio

$$\frac{e_0^0(m)}{e_1^0(m)} = \frac{\theta(b_m)}{\theta(b_m + \frac{1}{2})} \tag{104}$$

doesn't contain s and t. Thus from (48)(103) and (104) we have

$$\psi_{st}(u, Av) = \frac{f_{st}}{Af_{00}} = \frac{e^{2\pi i\phi_{st}}}{A} \frac{\theta(b_m + \frac{1}{2})e_0(st) + \theta(b_m)e_1(st)}{\theta(b_m + \frac{1}{2})e_0(00) + \theta(b_m)e_1(00)}, \tag{105}$$

where

$$e_j(st) = \theta(Av + \alpha_{st} + \frac{1}{2}j)\theta(Av + u + \beta_{st} + \frac{1}{2}j)\theta(u + \gamma_{st} + \frac{1}{2}j).$$

Let

$$\Theta(e^{-2\pi ix}) \equiv \theta(x, A\tau).$$

From (7) and (18)

$$P_{Z_6} = \sum_{s,t=0}^{A-1} u_1^s u_2^t e^{2\pi i\phi_{st}} \frac{\theta(b_m + \frac{1}{2})\varepsilon_0(st) + \theta(b_m)\varepsilon_1(st)}{A[\theta(b_m + \frac{1}{2})\varepsilon_0(00) + \theta(b_m)\varepsilon_1(00)]}, \tag{106}$$

where

$$\begin{aligned} \varepsilon_j(st) &= \theta(l(\tau_2\hat{y}_1 - \tau_1\hat{y}_2) + \alpha_{st} + \frac{i}{2}, A\tau) \times \theta(l((\tau_2 + 1)\,\hat{y}_1 - \tau_1\hat{y}_2) + \beta_{st} + \frac{j + A}{2}, A\tau) \\ &\quad \times\theta(l\hat{y}_2 + \gamma_{st} + \frac{j}{2}, A\tau) \\ &= \Theta(u_1^A e^{2\pi i\left(\alpha_{st} + \frac{1}{2}j\right)}) \times \Theta(u_2^A e^{2\pi i\left(\alpha_{st} + \frac{1}{2}j\right)}) \times \Theta(u_1^A u_2^A e^{2\pi i\left(\gamma_{st} + \frac{1}{2}j\right)}) \end{aligned} \tag{107}$$

Note that $\frac{A}{2}$ included in the second θ function in $\varepsilon_j(st)$ arise from

$$[l(\tau_2\hat{y}_1 - \tau_1\hat{y}_2), l\hat{y}_2] \equiv [A\hat{v}, \hat{u}] = il^2\tau_2 = 2\pi iA,$$

$$e^{A\hat{v}+\hat{u}} = e^{A\hat{v}}e^{\hat{u}}e^{-\pi iA} = e^{\hat{u}}e^{A\hat{v}}e^{\pi iA},$$

$$\begin{aligned} e^{A\hat{v}}e^{\hat{u}}\,|kq\rangle &= u_1^A u_2^A\,|kq\rangle = e^{Av+u}\,|kq\rangle \\ &= e^{A\hat{v}+\hat{u}+\pi iA}\,|kq\rangle\,. \end{aligned}$$

In (106),(107), the parameters $\alpha_{st}, \beta_{st}, \gamma_{st}$ are given by (80) and (81) with $B = 0$,

$$\alpha_{st} = [e^{\frac{\pi i}{6}}s + e^{-\frac{\pi i}{2}}t]\frac{\sqrt{3}}{3}, \quad (108)$$

$$\beta_{st} = e^{-\frac{\pi i}{3}}\alpha_{st} - \frac{A}{2}, \quad (109)$$

$$\gamma_{st} = e^{-\frac{2\pi i}{3}}\alpha_{st}, \quad (110)$$

$$\phi_{st} = \frac{\sqrt{3}i}{6A}(s^2 + t^2 - st) - \frac{st}{2A}, \quad (111)$$

$$b_m = \frac{1-A}{2} + m\frac{A}{2\tau - 1} = \frac{1-A}{2} - \frac{m}{3}A(2\tau - 1),\ m \neq 3p. \quad (112)$$

We take $m = 3p + M,\ M = \pm 1$ to obtain

$$\frac{\theta(b_m + \frac{1}{2})}{\theta(b_m)} = \frac{\theta(\frac{A}{2} - \frac{A(2\tau-1)}{3}, A\tau)}{\theta(\frac{A}{2} - \frac{1}{2} - \frac{A(2\tau-1)}{3}, A\tau)}, \quad (113)$$

which is independent of the choice of M, p. Now we check the covariance under transformation R. In the following, we find the expression (106) possesses manifest covariance. Actually, e_0 and e_1 are the covariant functions obtained from (81) by taking $B = 0$ and $B = 1$. Therefore they satisfy the covariant relation (83). We then check(75). From (105), the exponent of phase factor related to st on the left-handed side of (75) is proportional to

$$\phi_{st} - \frac{1}{2A}\left(t^2 - 2st\right) = \frac{\sqrt{3}i}{6A}\left(s^2 + t^2 - st\right) - \frac{1}{2A}\left(t^2 - st\right)$$

On the right-handed side phase factor is the exponent of

$$\begin{aligned} &\frac{\sqrt{3}i}{6A}\left((t-s)^2 + t^2 - t\,(t-s)\right) - \frac{1}{2A}\left(t^2 - st\right) \\ =\ &\frac{\sqrt{3}i}{6A}\left(s^2 + t^2 - st\right) - \frac{1}{2A}\left(t^2 - st\right) \end{aligned}$$

The two phase factors equal. From (83)(107), we know that the ψ_{st} given by (105) really satisfies the covariance relation (75). So (106) is the solution of projector which possesses manifest rotational covariance.

Now we have constructed the explicit and Manifestly covariant form for the projection operators on noncommutative integral orbifold T^2/Z_6 with trace $\frac{1}{A}$.

4 The general covariant projection operators

In this section we construct the general projectors with manifest covariant property by GHS construction. Instead of the vacuum $|0\rangle$, we take

$$|\phi_j\rangle = \int d^2z F_j(z)|z\rangle, \quad j=1,2 \tag{114}$$

where $|z\rangle$ is the coherent state satisfying the relation $a|z\rangle = \frac{l}{i\sqrt{2}}z|z\rangle$, $F_j(z)$ is an arbitrary continuous function of the argument z. Then for R in (12), we have

$$R|z\rangle = |e^{-i\theta}z\rangle \tag{115}$$

Now take $\theta = \frac{\pi}{3}, R = R_6$. When $F_j(z)$ satisfies the Z_6 symmetry

$$F_j(e^{\frac{\pi i}{3}}z) = e^{i\alpha_j}F_j(z), \tag{116}$$

we have

$$R|\phi_j\rangle = e^{i\alpha_j}|\phi_j\rangle. \tag{117}$$

Then we may obtain a projector in T^2/Z_6 from (40). In this case, we have

$$f_{st}(u,Av) = \int d^2z_1 d^2z_2 \sum_{r=0}^{A-1}\langle k, q_0 + \frac{l(r+s)}{A}|z_1\rangle\langle z_2|k, q_0+\frac{lr}{A}\rangle e^{2\pi i(\frac{q_0}{l}+\frac{r}{A})t}F_1(z_1)F_2(z_2)^*. \tag{118}$$

Define

$$G(u,v,z_1,z_2^*)_{ss'} \equiv \langle k, q_0+\frac{ls}{A}|z_1\rangle\langle z_2|k,q_0+\frac{ls'}{A}\rangle.$$

We have

$$f_{st}(u,Av) \equiv \int d^2z_1 d^2z_2 F_1(z_1)F_2(z_2)^* f_{st}(u,Av,z_1,z_2^*),$$

where

$$
\begin{aligned}
f_{st}(u, Av, z_1, z_2^*) &= \sum_{r=0}^{A-1} G(u, v, z_1, z_2^*)_{s+r,r} e^{2\pi i t(\frac{r}{A}+v)} \\
&= [c_0 \theta(Av + \alpha - Az_1\alpha_1 - Az_2^*\beta_1)\theta(Av + u + \beta - Az_1\beta_1 - Az_2^*\alpha_1) \\
&\quad \times \theta(u + \gamma - Az_1\gamma_1 + Az_2^*\gamma_1) + c_1\theta(Av + \alpha - Az_1\alpha_1 - Az_2^*\beta_1 + \frac{1}{2}) \\
&\quad \times \theta(Av + u + \beta - Az_1\beta_1 - Az_2^*\alpha_1 - \frac{1}{2})\theta(u + \gamma - Az_1\gamma_1 + Az_2^*\gamma_1 + \frac{1}{2})] \\
&\quad \times e^{2\pi i s(z_1 - z_2^*)\gamma_1 + 2\pi i t(z_1\alpha_1 + z_2^*\beta_1) + \frac{l^2}{4}(2z_1 z_2^* + |z_1|^2 + |z_2|^2)}. \qquad (119)
\end{aligned}
$$

The proof is given in Appendix B.

Due to (40), one has

$$
\psi_{st}(u, Av) = \frac{\int d^2z_1 d^2z_2 f_{st}(u, Av, z_1, z_2^*) F_1(z_1) F_2(z_2)^*}{A \int d^2z_1 d^2z_2 f_{00}(u, Av, z_1, z_2^*) F_1(z_1) F_2(z_2)^*}, \qquad (120)
$$

Therefore we obtain the explicit form for the general projection operators as follows,

$$
P = \frac{1}{A} \sum_{s,t=0}^{A-1} u_1^s u_2^t e^{2\pi i \phi_{st}} \frac{\theta(b_m + \frac{1}{2})\epsilon_0(st) + \theta(b_m)\epsilon_1(st)}{\theta(b_m + \frac{1}{2})\epsilon_0(00) + \theta(b_m)\epsilon_1(00)}, \qquad (121)
$$

where

$$
\epsilon_j(s, t) = \int dz_1 dz_2 F_1(z_1) F_2(z_2)^* E_j(s, t, z_1, z_2^*),
$$

and

$$
\begin{aligned}
E_j(s, t, z_1, z_2^*) &= \Theta\left(u_1^A e^{2\pi i(\alpha_{st} + \frac{j}{2})} e^{-2\pi i A(z_1\alpha_1 + z_2^*\beta_1)}\right) \times \Theta\left(u_1^A u_2^A e^{2\pi i(\beta_{st} + \frac{j}{2})} e^{-2\pi i A(z_1\beta_1 + z_2^*\alpha_1)}\right) \\
&\quad \times \Theta\left(u_2^A e^{2\pi i(\gamma_{st} + \frac{j}{2})} e^{-2\pi i A(z_1 - z_2^*)\gamma_1}\right). \qquad (122)
\end{aligned}
$$

When $F_j(z_j)$ satisfies Z_6 symmetry, namely in the case $\theta = \frac{\pi}{3}$ in formula (115), (121) shows the projectors P_{Z_6}, obviously the projector also belongs to P_{Z_3}. Just as proved in [40], we have gotten all the projectors with trace $\frac{1}{A}$ on the orbifolds T^2/Z_6 including the case that $\tilde{\psi}_{st}(k, q_0)$ is an analytic function. When $F_j(z_j)$ satisfies Z_3 symmetry but does not satisfy Z_6 symmetry, namely

$$
F_j(e^{\frac{2\pi i}{3}} z) = e^{i\alpha_j} F_j(z), \qquad (123)
$$

$$F_j(e^{\frac{\pi i}{3}}z) \neq const.F_j(z), \tag{124}$$

then (121) gives a projector of T^2/Z_3, but it is not a projector of T^2/Z_6. It is shown that the form of our solution possesses manifest covariance under rotation in Appendix B.

5 Discussion

We have found the complete set of projectors in analytic form with trace $\frac{1}{A}$ in all the cases of integral orbifold T^2/Z_N.(in the case of T^2/Z_4 refer to [40]), of course the case with trace $\frac{A-1}{A}$ is naturally obtained via the case with trace $\frac{1}{A}$ by $P' = id - P$.However we haven't obtained analytic solutions about projectors with an arbitrary trace $\frac{A-m}{A}, 1 < m < A-1$, which is an intriguing question that is closely related to the resolvent of the case that A is a rational number but not integer number. It is worthy of further study that whether there exists such an analytic solution or there is something special in its framework if such a solution exists.

Appendix A

Now we show briefly prove (87). Set $|\phi\rangle = |0\rangle$,

$$\begin{aligned} f_{st}(u, Av) &= \sum_{r=0}^{A-1} \langle k, q + \frac{l\,(s+r)}{A}|0\rangle\langle 0|k, q + \frac{lr}{A}\rangle \times e^{\frac{2\pi i t r}{A}} \times e^{\frac{2\pi i t q}{l}} \\ &= \frac{1}{l\sqrt{\pi}} \sum_r \theta\left(\tilde{v}_0 + \frac{\tilde{s}_0 + r}{A}, \frac{\tau}{A}\right) \theta\left(\tilde{v}_0 + \frac{r}{A}, -\frac{\tau^*}{A}\right) \\ &\quad \times e^{\frac{2\pi i t r}{A}} \times e^{\pi i \frac{\tau - \tau^*}{A} u^2 + 2\pi i \frac{s}{A} u + 2\pi i t v} \end{aligned} \tag{125}$$

Note $\tilde{v}_0 \neq v, \quad \tilde{s}_0 \neq s$ due to (84)(85) and $-\frac{\tau^*}{A} = \frac{\tau-1}{A}$. In terms of the definition of theta function, we expand the theta functions involved in F'_{st} defined as

$$F'_{st} = \sum_r \theta\left(\tilde{v}_0 + \frac{\tilde{s}_0 + r}{A}, \frac{\tau}{A}\right) \theta\left(\tilde{v}_0 + \frac{r}{A}, \frac{\tau - 1}{A}\right) e^{\frac{2\pi i t r}{A}} \tag{126}$$

in the Laurent series obtaining

$$\begin{aligned} F'_{st} &= \sum_r \left(\sum_m e^{\pi i \frac{\tau}{A} m^2} e^{2\pi i m \left(\tilde{v}_0 + \frac{\tilde{s}_0 + r}{A}\right)} \right) \\ &\times \left(\sum_{m'} e^{\pi i \frac{\tau - 1}{A} m'^2} e^{2\pi i m' \left(\tilde{v}_0 + \frac{r}{A}\right)} \right) \times e^{\frac{2\pi i t r}{A}}. \end{aligned}$$

After replacing variable m' by $n - m$, we get $F'_{st}(u, Av)$ as follow:

$$F'_{st} = \sum_{m,n} e^{\frac{\pi i}{A}\{m^2(2\tau-1)+n^2(\tau-1)-2mn(\tau-1)\}} \times e^{2\pi i n \tilde{v}_0} \times e^{2\pi i \frac{\tilde{s}_0}{A} m} \times \sum_{r=0}^{A-1} e^{2\pi i \frac{t+n}{A} r}.$$

Due to

$$\sum_{r=0}^{A-1} e^{2\pi i \frac{t+n}{A} r} = \begin{cases} A & \text{when } n = LA - t \quad L \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

and substituting $LA - t$ for n, (here L runs over all integers, after some computation and arrangement), we have

$$\begin{aligned} F'_{st} &= A \sum_{m,L} e^{\frac{\pi i}{A}(2\tau-1)\left\{\left(m-\frac{1}{2}(LA-t)\right)^2+\frac{1}{4}(LA-t)^2\right\}} \\ &\times e^{-\frac{\pi i}{2A}\left\{(LA)^2-2LAt\right\}} \times e^{-\frac{\pi i}{2A}t^2} \times e^{-\frac{\pi i}{A}\left(m-\frac{1}{2}(LA-t)\right)(t-2\tilde{s}_0)} \\ &\times e^{-\frac{\pi i}{A}\left(\frac{1}{2}(LA-t)\right)(t-2\tilde{s}_0)} \times e^{2\pi i (LA-t)\tilde{v}_0} \times e^{\pi i m L} \end{aligned}$$

Next we set $L = 2h + \delta$, here $h \in \mathbb{Z}, \delta = 0, 1$ and note the fact

$$e^{\pi i m L} = e^{\pi i m (2h+\delta)} = e^{\pi i m \delta}.$$

We obtain the form of sum over three variables N, m, δ as follows,

$$\begin{aligned} F'_{st} &= A \sum_{\delta=0,1} \sum_{m,h} e^{\pi i \left(\frac{2\tau-1}{A}\right)\left\{\left(m-\frac{1}{2}((2h+\delta)A-t)\right)^2+\frac{1}{4}((2h+\delta)A-t)^2\right\}} \qquad (127) \\ &\times e^{-\frac{\pi i}{2A}\left\{(2h+\delta)^2A^2-2(2h+\delta)At\right\}} \times e^{-\frac{\pi i}{2A}t^2} \times e^{\pi i m \delta} \times e^{2\pi i [(2h+\delta)A-t]\tilde{v}_0} \\ &\times e^{-\frac{\pi i}{A}\left(m-\frac{1}{2}((2h+\delta)A-t)\right)(t-2\tilde{s}_0)} \times e^{-\frac{\pi i}{2A}((2h+\delta)A-t)(t-2\tilde{s}_0)} \qquad (128) \end{aligned}$$

After arrangement, we find the sum over m and n can be separated into product of two theta functions. (note $\delta^2 = \delta$)

$$\begin{aligned}
F'_{st} &= A\sum_{\delta}\left(\sum_{h} e^{\pi i A(2\tau-1)h^2} \times e^{2\pi i h\left[\frac{\delta A - t + 2\tilde{s}_0}{2} + 2A\tilde{v}_0 + (2\tau-1)\frac{\delta A - t}{2}\right]}\right) &(129)\\
&\times\left(\sum_{m} e^{\pi i \frac{2\tau-1}{A}(m-hA)^2} \times e^{2\pi i (m-hA)\left[\frac{\delta A - t + 2\tilde{s}_0}{2A} - \left(\frac{2\tau-1}{A}\right)\left(\frac{\delta A - t}{2}\right)\right]}\right) &(130)\\
&\times e^{2\pi i \frac{2\tau-1}{A}\left(\frac{\delta A - t}{2}\right)^2} \times e^{-\frac{\pi i}{2A}(\delta A - t)^2} \times e^{2\pi i (\delta A - t)\tilde{v}_0} &(131)\\
&= \sum_{\delta=0,1} A\theta\left(\frac{(\delta A - t)}{A}(1-\tau) + \frac{\tilde{s}_0}{A}, \frac{2\tau-1}{A}\right) &(132)\\
&\times\theta\left((\delta A - t)\tau + \tilde{s}_0 + 2A\tilde{v}_0, A(2\tau-1)\right) \times e^{2\pi i \frac{2\tau-2}{A}\left(\frac{\delta A - t}{2}\right)^2} \times e^{2\pi i (\delta A - t)\tilde{v}_0} &(133)\\
&\equiv \sum_{\delta} A\theta\left(z, \frac{2\tau-1}{A}\right)\theta\left(w, A(2\tau-1)\right)e^{2\pi i\phi}, &(134)
\end{aligned}$$

where

$$\begin{aligned}
\phi &= \frac{\tau-1}{2A}(\delta A - t)^2 + (\delta A - t)\tilde{v}_0, &(135)\\
z &= \left(\frac{\delta A - t}{A}\right)(1-\tau) + \frac{\tilde{s}_0}{A}, &(136)\\
w &= (\delta A - t)\tau + \tilde{s}_0 + 2A\tilde{v}_0. &(137)
\end{aligned}$$

Having known that

$$f_{st} = F'_{st} e^{\pi i \frac{\tau-\tau^*}{A}u^2 + 2\pi i \frac{s}{A}u + 2\pi i t v},$$

from (125)(126) and $\tilde{v}_0 = v + \frac{\tau^* u}{A}$ and $\frac{\tilde{s}_0}{A} = \frac{s}{A} + \frac{(2\tau-1)u}{A}$, we obtain

$$f_{st} = \sum_{\delta} A\theta\left(z', \frac{2\tau-1}{A}\right)\theta\left(w, A(2\tau-1)\right)e^{2\pi i\phi'},$$

where

$$\begin{aligned}
z' &= z - t\frac{2\tau-1}{A} = -\delta\tau - \frac{\tau}{A}t + \frac{s}{A} + \frac{2\tau-1}{A}u,\\
w &= (\delta A - t)\tau + s + 2Av + u,\\
\phi' &= \delta A\frac{\tau-1}{2} + \delta(Av - (\tau-1)u) + \frac{2\tau-1}{2A}u^2 + \frac{s}{A}u,\\
&\quad -\frac{\tau}{A}tu - \frac{st}{A} + \frac{\tau}{2A}t^2.
\end{aligned}$$

This is the formula (87).

Appendix B

Now we would like to derive the general form of the projectors. We have the inner product of $< k, q|$ and the coherent state $|z>$ in paper [40]

$$< k, q|\,(z')>= \frac{1}{\sqrt{l}\pi^{1/4}}\theta(\frac{q+\frac{\tau}{\tau_2}k-i\sqrt{2}z'}{l},\frac{\tau}{A})e^{-\frac{\tau}{2i\tau_2}k^2+ikq+\sqrt{2}kz'-(z'^2+z'\bar{z}')/2},$$

where $a|\,(z')>=z'|\,(z')>$. Let $\frac{l}{i\sqrt{2}}z=z', |\,(z')>=|z>$, we have $a|z>=\frac{l}{i\sqrt{2}}z|z>$.(In [40], the coherent state is denoted by $|z>$ with $a|z>=z|z>$ which is the same as $|\,(z)>$ in this paper, we have given another implication to $|z>$ in this paper). Substitute the above formula into $G_{ss'}(u,v)$

$$\begin{aligned}
& G_{ss'}(u,v,z_1,z_2^*) \\
= & < k,q+\frac{ls}{A}|z_1>< z_2|k,q+\frac{ls'}{A}> \\
= & \frac{1}{l\sqrt{\pi}}\theta(v+\frac{\tau u}{A}+\frac{s}{A}-z_1,\frac{\tau}{A})\theta(v+\frac{\tau^* u}{A}+\frac{s'}{A}-z_2^*,\frac{-\tau^*}{A}) \\
& \times e^{\pi i\frac{\tau-\tau^*}{A}u^2+2\pi iu(\frac{s-s'}{A}-z_1+z_2^*)}\times e^{\frac{l^2}{4}(z_1^2+z_2^{*2}+z_1z_1^*+z_2z_2^*)}.
\end{aligned}$$

$$\begin{aligned}
f_{st}(u,Av,z_1,z_2^*) \equiv & \sum_{r=0}^{A-1}G_{s+r,r}(u,v,z_1,z_2^*)\times e^{2\pi it(\frac{r}{A}+v)} \\
= & \sum_r\frac{1}{l\sqrt{\pi}}\theta(v+\frac{\tau u}{A}+\frac{s+r}{A}-z_1,\frac{\tau}{A})\theta(v+\frac{r}{A}+\frac{u\tau^*}{A}-z_2^*,\frac{-\tau^*}{A})\times e^{2\pi itr/A} \\
& \times e^{\pi i\frac{\tau-\tau^*}{A}u^2+2\pi iu(\frac{s}{A}-z_1+z_2^*)+2\pi itv}\times e^{\frac{l^2}{4}(z_1^2+z_2^{*2}+z_1z_1^*+z_2z_2^*)}
\end{aligned}$$

Define $\tilde{v},\tilde{s}$ as

$$\tilde{v}+\frac{\tilde{s}}{A}=v+\frac{\tau u}{A}+\frac{s}{A}-z_1,\quad \tilde{v}=v+\frac{\tau^* u}{A}+z_2^* \tag{138}$$

From (138) we have for $\tau=e^{\frac{\pi i}{3}}$

$$\begin{aligned}
u = & \frac{-i}{\sqrt{3}}(\tilde{s}-s+A(z_1-z_2^*)) && (139)\\
Av = & A\tilde{v}+\frac{\tilde{s}-s}{2}(1+\frac{i}{\sqrt{3}})+\frac{Az_1}{2}(1+\frac{i}{\sqrt{3}})+\frac{Az_2^*}{2}(1-\frac{i}{\sqrt{3}}). && (140)
\end{aligned}$$

In terms of new variable $\tilde{v}$ and $\tilde{s}$,we have

$$
\begin{aligned}
f_{st}(u, Av, z_1, z_2^*) &= \frac{A}{l\sqrt{\pi}} \sum_\delta \theta((\delta A - t)(1-\tau) + \frac{\tilde{s}}{A}, \frac{2\tau-1}{A}) \\
&\times \theta((\delta A - t)\tau + 2A\tilde{v} + \tilde{s}, A(2\tau - 1) \times e^{2\pi i \frac{\tau-1}{2A}(\delta A - t)^2} \\
&\times e^{2\pi i \delta A \tilde{v}} \times e^{2\pi i t\left(\frac{\tilde{s}-s}{2A}(1+\frac{i}{\sqrt{3}}) + \frac{z_1}{2}(1+\frac{i}{\sqrt{3}}) + \frac{z_2^*}{2}(1-\frac{i}{\sqrt{3}})\right)} \\
&\times e^{\frac{\pi}{\sqrt{3}A}(\tilde{s}^2 - s^2 + 2sA(z_1 - z_2^*))} \times e^{\frac{l^2}{4}(2z_1 z_2^* + z_1 z_1^* + z_2 z_2^*)}.
\end{aligned} \tag{141}
$$

Taking $z_1 = z_2 = 0$ in Eq.(141) and denote $\tilde{v}_0 = \tilde{v}(z_1 = z_2 = 0), \tilde{s}_0 = \tilde{s}(z_1 = z_2 = 0)$ we get

$$
\begin{aligned}
f_{st}(u, Av, z_1, z_2^*)_{z_1=z_2=0} &= \frac{A}{l\sqrt{\pi}} \sum_\delta \theta\left((\delta A - t)(1-\tau) + \frac{\tilde{s}_0}{A}, \frac{2\tau-1}{A}\right) \\
&\quad \theta\left((\delta A - t)\tau + 2A\tilde{v}_0 + \tilde{s}_0, A(2\tau-1)\right) \\
&\quad \times e^{2\pi i \frac{\tau-1}{2A}(\delta A - t)^2} \times e^{2\pi i \delta A \tilde{v}_0} \times e^{2\pi i t \frac{\tilde{s}_0 - s}{2A}(1+\frac{i}{\sqrt{3}})} \times e^{\frac{\pi}{\sqrt{3}A}(\tilde{s}_0^2 - s^2)}.
\end{aligned} \tag{142}
$$

On the other hand, when $z_1 = z_2 = 0$, $f_{st}(u, Av, 0, 0) = f_{st}(u, Av)$. We have from (84)(90)(101)

$$
\begin{aligned}
f_{st}(u, Av) &= c_0 \theta\left(A\tilde{v}_0 + \frac{\tilde{s}_0 - s}{2}(1 + \frac{i}{\sqrt{3}}) + \alpha\right) \\
&\times \theta\left(A\tilde{v}_0 + \frac{\tilde{s}_0 - s}{2}(1 - \frac{i}{\sqrt{3}}) + \beta\right) \theta\left(\frac{-i}{\sqrt{3}}(\tilde{s}_0 - s) + \gamma\right) \\
&+ c_1 \theta\left(A\tilde{v}_0 + \frac{\tilde{s}_0 - s}{2}(1 + \frac{i}{\sqrt{3}}) + \alpha + \frac{1}{2}\right) \\
&\times \theta\left(A\tilde{v}_0 + \frac{\tilde{s}_0 - s}{2}(1 - \frac{i}{\sqrt{3}}) + \beta - \frac{1}{2}\right) \theta\left(\frac{-i}{\sqrt{3}}(\tilde{s}_0 - s) + \gamma + \frac{1}{2}\right),
\end{aligned} \tag{143}
$$

where we have from (96) (102)

$$
\begin{aligned}
c_0 &= \frac{A}{l\sqrt{\pi}} e^{2\pi i(\phi_{st} + \phi_{\delta=0})} \theta\left(\delta \frac{(1-A)}{2} \frac{2\tau-1}{A}, \frac{2\tau-1}{A}\right) \\
&\times \theta\left((\frac{1}{2} - \frac{m}{3})A(2\tau-1) + \frac{1}{2}, A(2\tau-1)\right) \\
&\div \left\{\theta\left(\frac{A\tau+1}{2} + \frac{1}{2}\right) \theta\left(\frac{A\tau+1}{2} - \frac{m}{3}A(2\tau-1), A\tau\right) \theta\left(\frac{1-A}{2} - \frac{m}{3}A(2\tau-1)\right)\right\},
\end{aligned} \tag{144}
$$

$$\begin{aligned}
c_1 &= \frac{A}{l\sqrt{\pi}} e^{2\pi i(\phi_{st}+\phi_{\delta=1})} \theta\left(\delta\frac{(1-A)}{2}\frac{2\tau-1}{A}, \frac{2\tau-1}{A}\right) \\
&\quad \times\theta\left((\frac{1}{2}-\frac{m}{3})A(2\tau-1)+\frac{1}{2}, A(2\tau-1)\right) \\
&\quad \div\left\{\theta\left(\frac{A\tau+1}{2}+\frac{1}{2}\right)\theta\left(\frac{A\tau+1}{2}-\frac{m}{3}A(2\tau-1), A\tau\right)\theta\left(\frac{1-A}{2}-\frac{m}{3}A(2\tau-1)+\frac{1}{2}\right)\right\}.
\end{aligned} \tag{145}$$

Thus as an analytic function of $A\tilde{v}_0$ and $\tilde{s}_0$, the right-handed side of (142) equals to the right-handed side of (143) holds for all u and v (see section 3) and it is an identity. Through the transformation of $A\tilde{v}_0 \to A\tilde{v}$, $\tilde{s}_0 \to \tilde{s}$, we get

$$\begin{aligned}
& \frac{A}{l\sqrt{\pi}}\sum_{\delta}\theta\left((\delta A-t)(1-\tau)+\frac{\tilde{s}}{A}, \frac{2\tau-1}{A}\right) \\
& \theta\left((\delta A-t)\tau+2A\tilde{v}+\tilde{s}, A(2\tau-1)\right. \\
& \times e^{2\pi i\frac{\tau-1}{2A}(\delta A-t)^2}\times e^{2\pi i\delta A\tilde{v}}\times e^{2\pi i t\frac{\tilde{s}-s}{2A}(1+\frac{i}{\sqrt{3}})}\times e^{\frac{\pi}{\sqrt{3}A}(\tilde{s}^2-s^2)} \\
= \; & c_0\theta\left(A\tilde{v}+\frac{\tilde{s}-s}{2}(1+\frac{i}{\sqrt{3}})+\alpha\right)\theta\left(A\tilde{v}+\frac{\tilde{s}-s}{2}(1-\frac{i}{\sqrt{3}})+\beta\right)\theta\left(-\frac{i}{\sqrt{3}}\frac{\tilde{s}-s}{2}+\gamma\right) \\
& +c_1\theta\left(A\tilde{v}+\frac{\tilde{s}-s}{2}(1+\frac{i}{\sqrt{3}})+\alpha+\frac{1}{2}\right)\theta\left(A\tilde{v}+\frac{\tilde{s}-s}{2}(1-\frac{i}{\sqrt{3}})+\beta-\frac{1}{2}\right) \\
& \times\theta\left(-\frac{i}{\sqrt{3}}\frac{\tilde{s}-s}{2}+\gamma+\frac{1}{2}\right)\equiv f'_{st}.
\end{aligned} \tag{146}$$

Observing the right-hand side of (141) compared with the left-hand side of (146), it is easy to find

$$\begin{aligned}
f_{st}(u, Av, z_1, z_2^*) &= f'_{st}\times e^{2\pi i t[(z_1(\frac{1}{2}+\frac{\sqrt{3}i}{6})+z_2^*(\frac{1}{2}-\frac{\sqrt{3}i}{6})]}\times e^{2\pi i s(\frac{-\sqrt{3}i}{3}z_1+\frac{\sqrt{3}i}{3}z_2^*)} \\
&\quad \times e^{\frac{l^2}{4}(2z_1z_2^*+z_1z_1^*+z_2z_2^*)}
\end{aligned}$$

Substituting (138) into (146), we get

$$
\begin{aligned}
f_{st}(u, Av, z_1, z_2^*) \quad = \quad & c_0\theta\left(Av - Az_1(\frac{1}{2} + \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} - \frac{\sqrt{3}i}{6}) + \alpha\right) \\
& \times\theta\left(Av + u - Az_1(\frac{1}{2} - \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} + \frac{\sqrt{3}i}{6}) + \beta\right) \\
& \times\theta\left(u - Az_1(-\frac{i}{\sqrt{3}}) - Az_2^*(\frac{i}{\sqrt{3}}) + \gamma\right) \\
& +c_1\theta\left(Av - Az_1(\frac{i}{\sqrt{3}}) - Az_2^*(\frac{1}{2} - \frac{\sqrt{3}i}{6}) + \alpha + \frac{1}{2}\right) \\
& \times\theta\left(Av + u - Az_1(\frac{1}{2} - \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{i}{\sqrt{3}}) + \beta - \frac{1}{2}\right) \\
& \times\theta\left(u - Az_1(-\frac{i}{\sqrt{3}}) - Az_2^*(\frac{i}{\sqrt{3}}) + \gamma + \frac{1}{2}\right) \\
& \times e^{2\pi it[(z_1(\frac{1}{2}+\frac{\sqrt{3}i}{6})+z_2^*(\frac{1}{2}-\frac{\sqrt{3}i}{6})]} \times e^{2\pi is(\frac{-\sqrt{3}i}{3}z_1+\frac{\sqrt{3}i}{3}z_2^*)} \times e^{\frac{l^2}{4}(2z_1z_2^*+z_1z_1^*+z_2z_2^*)} \\
& . \qquad (147)
\end{aligned}
$$

Take the transformation

$$u \longrightarrow -Av, \quad Av \longrightarrow -\frac{A}{2} + u + Av.$$

The f_{st} is transformed into $\bar{f}_{st}$

$$\begin{aligned}
\bar{f}_{st}(u, Av, z_1, z_2^*) &= c_0(s,t)\theta\left(Av - Az_1(\frac{i}{\sqrt{3}}) - Az_2^*(-\frac{i}{\sqrt{3}}) + \alpha_{t,t-s}\right) \\
&\times\theta\left(Av + u - Az_1(\frac{1}{2} + \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} - \frac{\sqrt{3}i}{6}) + \beta_{t,t-s}\right) \\
&\times\theta\left(u - Az_1(\frac{1}{2} - \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} + \frac{\sqrt{3}i}{6}) + \gamma_{t,t-s}\right) \\
&+c_1(s,t)\theta\left(Av - Az_1(\frac{i}{\sqrt{3}}) - Az_2^*(-\frac{i}{\sqrt{3}}) + \alpha_{t,t-s} + \frac{1}{2}\right) \\
&\times\theta\left(Av + u - Az_1(\frac{1}{2} + \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} - \frac{\sqrt{3}i}{6}) + \beta_{t,t-s} - \frac{1}{2}\right) \\
&\times\theta\left(u - Az_1(\frac{1}{2} - \frac{\sqrt{3}i}{6}) - Az_2^*(\frac{1}{2} + \frac{\sqrt{3}i}{6}) + \gamma_{t,t-s} + \frac{1}{2}\right) \\
&\times e^{2\pi i(t-s)(\frac{\sqrt{3}i}{3}z_1 - \frac{\sqrt{3}i}{3}z_2^*)} \times e^{2\pi it[(z_1(\frac{1}{2}-\frac{\sqrt{3}i}{6})+z_2^*(\frac{1}{2}+\frac{\sqrt{3}i}{6})]} \\
&\times e^{\frac{l^2}{4}(2z_1z_2^*+z_1z_1^*+z_2z_2^*)}. \qquad (148)
\end{aligned}$$

Again we let $s' = t, t' = t-s$ and change z_1 and z_2 into $z_1' = e^{\frac{2\pi i}{6}}z_1$ and $z_2' = e^{-\frac{2\pi i}{6}}z_2$ and rewrite $c_j(st)$ as $e^{2\pi i\phi_{st}}c_j(00)$.(see (144) and (145)and the definition of ϕ_{st}, ϕ_δ in (95)) We have

$$
\begin{aligned}
\bar{f}_{st}(u,Av,z_1,z_2^*) &= e^{2\pi i\phi_{st}}c_0(0,0)\theta\left(Av - Az_1'(\frac{1}{2}+\frac{\sqrt{3}i}{6}) - Az_2'^*(\frac{1}{2}-\frac{\sqrt{3}i}{6}) + \alpha_{s',t'}\right)\\
&\times\theta\left(Av + u - Az_1'(\frac{1}{2}-\frac{\sqrt{3}i}{6}) - Az_2'^*(\frac{1}{2}+\frac{\sqrt{3}i}{6}) + \beta_{s',t'}\right)\\
&\times\theta\left(u - Az_1'(-\frac{i}{\sqrt{3}}) - Az_2'^*(\frac{i}{\sqrt{3}}) + \gamma_{s',t'}\right)\\
&+e^{2\pi i\phi_{st}}c_1(0,0)\theta\left(Av - Az_1'(\frac{1}{2}+\frac{\sqrt{3}i}{6}) - Az_2'^*(\frac{1}{2}-\frac{\sqrt{3}i}{6}) + \alpha_{s',t'}) + \frac{1}{2}\right)\\
&\times\theta\left(Av + u - Az_1'(\frac{1}{2}-\frac{\sqrt{3}i}{6}) - Az_2'^*(\frac{1}{2}+\frac{\sqrt{3}i}{6}) + \beta_{s',t'} - \frac{1}{2}\right)\\
&\times\theta\left(u - Az_1'(-\frac{i}{\sqrt{3}}) - Az_2'^*(\frac{i}{\sqrt{3}}) + \gamma_{s',t'} + \frac{1}{2}\right)\\
&\times e^{2\pi it'[(z_1'(\frac{1}{2}+\frac{\sqrt{3}i}{6})+z_2'^*(\frac{1}{2}-\frac{\sqrt{3}i}{6})]} \times e^{\frac{l^2}{4}(2z_1'z_2'^*+z_1'z_1'^*+z_2'z_2'^*)} \times e^{2\pi is'(\frac{-\sqrt{3}i}{3}z_1'+\frac{\sqrt{3}i}{3}z_2'^*)}.
\end{aligned}
\tag{149}
$$

It is easy to check that

$$
e^{2\pi i\phi_{st}} = c^{2st-t^2}e^{2\pi i\phi_{s',t'}},
$$

where

$$
\phi_{st} = \frac{\sqrt{3}i}{6A}\left(s^2+t^2-st\right) - \frac{1}{2A}st.
$$

Therefore, we have

$$
f_{s't'}(u,Av,z_1',z_2'^*) = c^{-2st+t^2}\bar{f}_{st}(u,Av,z_1,z_2^*). \tag{150}
$$

Finally we check the $\psi_{s,t}(u,Av)$ constructed from f_{st} for the covariant condition (75),

$$
c^{-2st+t^2}\psi_{st}(-Av,-\frac{A}{2}+u+Av) = \psi_{t,t-s}(u,Av) = \psi_{s',t'}(u,Av). \tag{151}
$$

Since

$$
\begin{aligned}
\psi_{s',t'}(u,Av) &= \frac{f_{s't'}(u,v)}{Af_{00}(u,v)}\\
&= \frac{\int d^2z_1'd^2z_2'f_{s't'}(u,Av,z_1',z_2'^*)F_1(z_1')F_2(z_2')^*}{A\int d^2z_1'd^2z_2'f_{00}(u,Av,z_1',z_2'^*)F_1(z_1')F_2(z_2')^*}\\
&= \frac{c^{-2st+t^2}\int d^2z_1'd^2z_2'\bar{f}_{st}(u,Av,z_1,z_2^*)F_1(z_1')F_2(z_2')^*}{A\int d^2z_1'd^2z_2'\bar{f}_{00}(u,Av,z_1,z_2^*)F_1(z_1')F_2(z_2')^*}
\end{aligned}
$$

Based on (116) and (120)

$$\begin{aligned}
&\psi_{st}(-Av, -\frac{A}{2} + u + Av) \\
= \quad &\frac{\int d^2z_1 d^2z_2 \bar{f}_{st}(u, Av, z_1, z_2^*) F_1(z_1) F_2(z_2^*)}{A \int d^2z_1 d^2z_2 \bar{f}_{00}(u, Av, z_1, z_2^*) F_1(z_1) F_2(z_2^*)}.
\end{aligned}$$

It is obvious that the equation (151) holds.

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